American Style Securities

An American Style Security can be exercised at 'any' time up to its expiry. As we have discussed previously, this allows exercise at any stopping time of the filtration. A discussion of this kind of security in continuous time involves several techical issues. We can avoid some of these by assuming our time set is discrete and finite, so our discussion is limited to this case.

Our stochastic base will be

$$(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_n), \{0,1,2,..,T\})$$

and when we speak of processes they will be finite sequences;

$$X_0, X_1, \ldots, X_T$$

with X_i an $\mathcal{F}_i$ measurable random variable. A stopping time in this context is a map

$$\tau : \Omega \longrightarrow \{0,1,2,..,T\}$$

with $\{\tau \leq i\} \in \mathcal{F}_i$ or, equivalently,

$$\{\tau = i\} \in \mathcal{F}_i \quad , 0 \leq i \leq T.$$

An American style security is a non-negative process (X_i) and the risk-less bond is the (deterministic) process

$$B_i = (1+r)^i, \quad r > 0$$

We will assume that there is a risky asset $S \equiv (S_i)$ which we think of as a stock and much of our discussion will be about S but will apply to a general process such as (X_i). The call option provides an example of the American style security: the payoff at time i is $(S_i - K)^+$ and it can be exercised at each $i \in \{0, 1, 2, \ldots, T\}$. But more is true, it can be exercised at any stopping time τ with a payoff of $(S_\tau - K)^+$. This allows differing scales of (random) payoff: For example; τ_1 = first time S hits $K+1$ or T, if it fails to hit $K+1$, and τ_2 = first time S hits $K+2$ or T, if it fails to hit $K+2$, provide distinct random payoffs.

Suppose you have written (sold) an American Call on S. Your liability is to meet the payoff whatever time of exercise is chosen by the holder of the option. So you must be able to meet $(S_\tau - K)^+$ for any τ. This presents a problem. We know how to find a replicating portfolio for a single payoff, say, $(S_{\tau_0}-K)^+$, for a particular τ_0. But how can we hedge a whole collection of payoffs simultaneously? We investigate. First of all we assume that we can replicate all claims in our discrete time model, that $\mathbb{P}$ is a risk-neutral measure for B, so discounted portfolios in S and B are $\mathbb{P}$-martingale. The notion of a self-financing portfolio is exactly as before save that our stochastic integrals are now finite sums,

$$\int_0^t \phi_s dS_s = \sum_{i=1}^{t} \phi_{i-1} \Delta S_i \qquad (\Delta S_i = S_i - S_{i-1})$$

(remark: the self-financing condition now reads $\psi_{i-1} B_i + \phi_{i-1} S_i = \psi_i B_i + \phi_i S_i$)

So suppose we have a self-financing portfolio, $V = \phi S + \psi B$, then $\tilde{V} = V/B$ is a $\mathbb{P}$-martingale and let us suppose we are trying to hedge the liability $(S_\tau - K)^+$, where for now τ is fixed. We will assume that once exercise has occured the proceeds are invested in the bond until time T. So, first of all, on the set $\{\tau = i\}$ the payoff is $(S_i - K)^+$, that is,

$$(S_\tau - K)^+ = \Big(\sum_{i=0}^{T} S_i I_{\{\tau=i\}} - K \Big)^+$$

$$= \sum_{i=0}^{T} (S_i - K)^+ I_{\{\tau=i\}}$$

at time T these amounts become $(S_i - K)(1+r)^{T-i}$ on $\{\tau = i\}$, that is,

$$\sum_{i=0}^{T} (S_i - K)^+ (1+r)^{T-i} I_{\{\tau=i\}}$$

So this is the time T value of our payoff. The time $t=0$ value of this we know to be

$$\mathbb{E}^{\mathbb{P}} \left(\frac{\sum_{i=0}^{T} (S_i - K)^+ (1+r)^{T-i} I_{\{\tau=i\}}}{(1+r)^T} \right)$$

by the usual risk-neutral pricing formula. This is

$$\mathbb{E}^{\mathbb{P}} \Big(\sum_{i=0}^{T} (S_i - K)^+ (1+r)^{-i} I_{\{\tau=i\}} \Big)$$

$$= \mathbb{E}^{\mathbb{P}} \left(\Big(\sum_{i=0}^{T} (S_i - K)^+ I_{\{\tau=i\}} \Big) \cdot \Big(\sum_{j=0}^{T} (1+r)^{-j} I_{\{\tau=j\}} \Big) \right)$$

$$= \mathbb{E}^{\mathbb{P}}\left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right)$$ and this is the

inital endowment required to hedge the payoff $(S_\tau - K)^+$. If there is a self-financing portfolio, (ϕ^*, ψ^*), which can be a hedge for every τ then, because there is no arbitrage,

$$V_0(\phi^*, \psi^*) \geq \sup_\tau \mathbb{E}^{\mathbb{P}}\left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right).$$

This is a necessary condition for the inital value of a hedge for the American Call. It is difficult to see where to go next until one asks a question: Is there a stopping time, τ^*, such that

$$\mathbb{E}^{\mathbb{P}}\left(\frac{(S_{\tau^*} - K)^+}{(1+r)^{\tau^*}} \right) = \sup_\tau \mathbb{E}^{\mathbb{P}}\left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right)?$$

It turns out that the answer is yes! The time τ^* is called "optimal" for this problem and it is intimately tied up with the nature of the hedge for this security.

Snell's Envelope

This is a quite general piece of mathematics and we take advantage of this to widen the discussion to an American security X which is such that

$$X_i = f_i(S_i) \qquad i \in \{0,1,2,\ldots,T\}.$$

So at time i the payoff, X_i, is some function, f_i, of S_i. For the call option,

$$X_i = f(S_i) = (S_i - K)^+$$

so that $f_0(x) = f_1(x) = \ldots = f_T(x) = (x-K)^+.$

We consider time T. Here the payoff is X_T. So at time T our hedge should be at least X_T. (†) Moving to time $T-1$ the situation is a little different. There are two courses of action. Exercise or continuation. The time $T-1$ value of exercise is, of course, X_{T-1}. The time $T-1$ value of continuing to hold the security is the time $T-1$ value of X_T. We know how to calculate this; all discounted portfolios are $\mathbb{P}$ martingales so writing C_i for the time i value of X_T ⊛

So $V_T = X_T$

use t as time since we start with T

$$M_i^{\mathbb{P}}\left(\frac{X_T}{B_T}\right) = \frac{C_i}{B_i}$$

that is,

$$C_i = B_i M_i^{\mathbb{P}}\left(\frac{X_T}{B_T}\right)$$

So the time $T-1$ value of continuation, C_{T-1}, is

$$C_{T-1} = M^{\mathbb{P}}_{T-1}\left(\frac{X_T}{1+r}\right)$$

⊛ REM The time $T-1$ payoff of the security, X_{T-1}, and the time $T-1$ value of X_T are in general <u>not</u> the same.

So our hedge must be greater than X_{T-1} and $M^{\mathbb{P}}_{T-1}\left(\frac{X_T}{1+r}\right)$, in short,

$$V_{T-1} \geq X_{T-1} \vee M^{\mathbb{P}}_{T-1}\left(\frac{X_T}{1+r}\right)$$

and the right side is the value of the American security at time $T-1$. Now we move to time $T-2$. Again we have two courses of action: exercise or continuation. The value of exercise is X_{T-2} while the value of continuation is the time $T-2$ value of the time $T-1$ value of the security, this last is

$$M^{\mathbb{P}}_{T-2}\left(\frac{X_{T-1} \vee M^{\mathbb{P}}_{T-1}\left(\frac{X_T}{1+r}\right)}{1+r}\right).$$

So the hedge at time $T-2$ must satisfy,

$$V_{T-2} \geq X_{T-2} \vee M^{\mathbb{P}}_{T-2}\left(\frac{X_{T-1} \vee M^{\mathbb{P}}_{T-1}\left(\frac{X_T}{1+r}\right)}{1+r}\right).$$

The right side is the time T-2 value of the security. We need to introduce some notation to make the situation clearer! Let us write

$$Z^0_T = X_T$$

$$Z^0_t = X_t \vee M^{\mathbb{P}}_t\left(\frac{Z^0_{t+1}}{1+r}\right) \quad t<T$$

There is an obvious pattern here but moving to discounted values makes the matter even clearer,

$$Z_T = \frac{Z^0_T}{B_T} = \frac{X_T}{B_T}$$

$$Z_t = \frac{Z^0_t}{B_t} = \left(\frac{X_t}{B_t}\right) \vee M^{\mathbb{P}}_t\left(\frac{Z^0_{t+1}}{B_t(1+r)}\right)$$

$$= \left(\frac{X_t}{B_t}\right) \vee M^{\mathbb{P}}_t\left(Z_{t+1}\right)$$

So our hedging problem is to construct a portfolio whose discounted values track (Z_t) or better, i.e.,

$$\tilde{V}_t \geq Z_t.$$

The construction of the sequence (Z_t) is very general. All we assume is that (X_t) is a non-negative adapted sequence and our argument constructs Z, the Snell Envelope of $(\tilde{X}_t)$. Note that although we explicitly employed discounting to arrive at Z we can define the Snell Envelope for *any* non-negative adapted sequence (Y_t). One simply defines

$$Z^Y_T = Y_T$$

$$Z^Y_{T-s} = Y_{T-s} \vee M^{\mathbb{P}}_{T-s}(Z^Y_{T-s+1})$$

for $T \geq s \geq 0$.

The process Z has lots of remarkable properties.

<u>Theorem</u>

Let Z and $\tilde{X}$ be as above.

(i) Z is the least supermartingale dominating $\tilde{X}$.

(ii) Define $\tau^*(\omega) = \min\{t \geq 0 : Z_t = \tilde{X}_t\}$. Then τ^*, the first time $\tilde{X}$ is equal to Z, is a stopping time. Moreover, $Z^{\tau^*} = (Z_{\tau^* \wedge t})$ is a $\mathbb{P}$ martingale.

Pf

(i) By definition $Z_{T-s} = \tilde{X}_{T-s} \vee M^{\mathbb{P}}_{T-s}(Z_{T-s+1})$

$$\geq \tilde{X}_{T-s}, \quad s>0$$

and $Z_T = \tilde{X}_T$. So Z dominates $\tilde{X}$. Also,

$$Z_{T-s} \geq M^{\mathbb{P}}_{T-s}(Z_{T-s+1}) \quad s>0, \text{etc},$$

which is enough to show Z is a supermartingale. Suppose that $Y=(Y_t)$ is a supermartingale with $Y_t \geq \tilde{X}_t$. Then, $Y_T \geq \tilde{X}_T = Z_T$. If for any t we have $Y_t \geq Z_t$ then as Y is a supermartingale,

$$Y_{t-1} \geq M^{\mathbb{P}}_{t-1}(Y_t) \geq M^{\mathbb{P}}_{t-1}(Z_t)$$

while $\quad Y_{t-1} \geq \tilde{X}_{t-1}$. So

$$Y_{t-1} \geq \tilde{X}_{t-1} \vee M^{\mathbb{P}}_{t-1}(Z_t) = Z_{t-1}$$

(proof by induction)

(ii) By definition $\{\tau^* = 0\} = \{X_0 = Z_0\} \in \mathcal{F}_0$ while

$$\{\tau^* = t\} = \left[\bigcap_{i=0}^{t-1} \{\tilde{X}_i < Z_i\}\right] \cap \{Z_t = \tilde{X}_t\} \in \mathcal{F}_t$$

(these are all $\mathcal{F}_t$ sets). Note also that since $Z_T = \tilde{X}_T$, $\tau^* \leq T$.

Consider

In fact prove more

$$Z_{\tau^*\wedge t} = \sum_{i\le t} Z_i I_{\{\tau^*=i\}} + Z_t I_{\{\tau^*>t\}}$$

exercise, finish this proof.

Def^n/ A stopping time, σ, will be called optimal for $\tilde{X}$ if

$$\mathbb{E}(\tilde{X}_\sigma) = \sup_\tau \mathbb{E}(\tilde{X}_\tau)$$

Thm The time τ^* is optimal for $\tilde{X}$.

Pf Z^{τ^*} is a martingale and, as usual, $\mathcal{F}_0$ is the trivial σ-field. So $\tilde{X}_0$ and Z_0 are constants (a.s.). Therefore

$$Z_0 = Z_{\tau^*\wedge 0} = \mathbb{E}(Z_{\tau^*\wedge T}) = \mathbb{E}(Z_{\tau^*})$$

$= \mathbb{E}(\tilde{X}_{\tau^*})$ (because τ^* is the first time $\tilde{X} = Z$). But Z is a supermartingale, so for any time, σ, Z^σ is a supermartingale, and,

$$Z_0 = \mathbb{E}(Z_{\sigma\wedge 0}) \ge \mathbb{E}(Z_\sigma) \ge \mathbb{E}(\tilde{X}_\sigma)$$

because Z dominates $\tilde{X}$. So

$$\mathbb{E}(\tilde{X}_{\tau^*}) = Z_0 = \sup_\sigma \mathbb{E}(\tilde{X}_\sigma)$$

$\square$

It turns out that τ^* provides the optimal time of exercise of the American security: an heuristic argument follows. Since τ^* is the first time $Z = \tilde{X}$ then on the set $\{\tau^* = t\}$ we must have $Z_i \neq \tilde{X}_i$ for $0 \leq i \leq t-1$ (with nothing to say if $t = 0$... but see later!). Since $Z_i = \tilde{X}_i \vee M_i(Z_{i+1})$ it has to be that $Z_i = M_i(Z_{i+1}) > \tilde{X}_i$ for $0 \leq i \leq t-1$ on the set $\{\tau^* = t\}$. Put another way; continuation has a greater value than exercise. Because we "prefer more to less" the "rational" thing to do is to choose continuation rather than exercise on $\{\tau^* = t\}$ prior to t. At time t, $\tilde{X}_t = Z_t$, that is, exercise and continuation have the same value on $\{\tau^* = t\}$. So one can choose either "option" at this point. But look ahead. Since Z is a supermartingale

$$
\begin{aligned}
M_t(Z_{t+s} I_{\{\tau^* = t\}}) &= M_t(Z_{t+s}) I_{\{\tau^* = t\}} \\
&\leq Z_t I_{\{\tau^* = t\}} \\
&= \tilde{X}_t I_{\{\tau^* = t\}}
\end{aligned}
$$

But

$$Z_{t+s} = \tilde{X}_{t+s} \vee M_{t+s}(Z_{t+s+1}) \geq \tilde{X}_{t+s}$$

so

$$I_{\{\tau^*=t\}} M_t(\tilde{X}_{t+s}) \leq I_{\{\tau^*=t\}} M_t(Z_{t+s}) \leq Z_t I_{\{\tau^*=t\}}$$

$$= \tilde{X}_t I_{\{\tau^*=t\}}$$

So,

$$I_{\{\tau^*=t\}} B_t M_t(\tilde{X}_{t+s}) \leq B_t \tilde{X}_t I_{\{\tau^*=t\}}$$

which 'states' that the time t value of exercise at time $t+s$ is less than the (time t) value of exercise at time t all on the set $\{\tau^*=t\}$. So you should exercise at time t on the set $\{\tau^*=t\}$.

Of course this argument is a mixture of finance and mathematics. A rigorous treatment requires more time than we have.

To find a hedge for the American Security we use martingale representation to make a (discounted) portfolio which tracks $Z^{\tau^*} = (Z_{\tau^* \wedge t})$. There is a lot more to say of this but we do not have time.

The relationship between the American Call and the European Call can be derived from what we have done above.

The discounted value process for the American call is given by (Z_t). For the case $X_t = (S_t - K)^+$. Let $(\bar{C}_t)$ be the discounted value process for the European Call. Note that

$$\bar{C}_T = \tilde{X}_T = Z_T$$

Now

$$Z_t \geq M_t(Z_T) = M_t(\tilde{X}_T) = M_t(\bar{C}_T) = \bar{C}_t$$

because Z is a supermartingale and $(\bar{C}_t)$ is a $\mathbb{P}$ martingale. So $Z_t \geq \bar{C}_t$.
On the other hand Call-Put parity tells us that, for European Options,

$$C_t - P_t = S_t - \frac{K}{e^{r(T-t)}}$$
$$\geq S_t - K$$

So that $C_t \geq S_t - K$. However $C_t \geq 0$ everywhere so it must be that $C_t \geq (S_t - K)^+$. It follows that

$$\bar{C}_t \geq \tilde{X}_t$$

and $(\bar{C}_t)$ is a $\mathbb{P}$-martingale and therefore a $\mathbb{P}$-supermartingale and note, it dominates $(\tilde{X}_t)$. So $\bar{C}_t \geq Z_t$ and we have, $\bar{C}_t = Z_t$ for all t. The value processes agree! One can further show that $\tau^* = T$. (Exercise?).